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Spinor Strong Interaction Model for Meson Spectra

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A model for a bound quark–antiquark system is constructed from quark spinor equations and the associated pseudoscalar massless interaction potential equations in a way departing from conventional relativistic quantum mechanics. From the so-constructed covariant meson equations, linear confinement arises naturally. Nonlinear radial equations for the pseudoscalar and vector mesons in the rest frame are derived without approximation. An internal complex space is introduced for representation of the quark flavors. Quark masses are generalized to operators operating on functions in this space. A simple model is proposed for the meson internal functions and mass operators producing the squares of the average quark masses as eigenvalues. The present space-time model calls for a particle classification scheme different from the usual nonrelativistic one. When combined with the internal model, it may account for the gross structure of the meson spectra together with the form of an empirical relation. Upper limits of bare quark masses are estimated from simplified analytical solutions of the radial equations and agree approximately with the bare quark masses obtained from baryon data in a companion paper. The radial equations are solved numerically yielding estimates of the strong interaction radii of the ground state mesons.

1. INTRODUCTION

In the many attempts to account for the hadron spectra in the years after the proposal of the quark hypothesis of Gell-Mann and Zweig during the early 1960s, the Bethe–Salpeter (BS) equation, in its various simplified forms, has naturally emerged as the dominant starting point. In the mid-1970s, quantum chromodynamics (QCD) was advocated, in the wake of the advance of the electroweak gauge model of Weinberg and Salam, as the theory for strong interactions (e.g., Particle Data Group, 1990). Following

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this development, a hadron spectra model was proposed (de Rújula *et al.*, 1975) which includes some essential phenomenological and unique features of QCD but is still based upon starting points which may be regarded as some coarse approximation of the BS equation. This and related approaches have been pursued and improved upon by many authors, notably Isgur and Mitra [e.g., see references in Lichtenberg (1987)], resulting in a large body of literature, part of which has been reviewed by Lichtenberg (1987).

Many of these QCD-oriented models yield good and encompassing but parameter-dependent and incomplete agreement with data. These have, however, among numerous difficulties, two main ones. In the first place, the confining potentials employed are in principle *ad hoc* assumptions, as QCD has not been proven to be confining. Second, the BS equation has been simplified in so many ways that it is not possible to estimate adequately the consequences of the various approximations, notably departure from relativity. It is therefore desirable to replace the QCD-oriented BS equation approach by one free from these difficulties.

The purpose of this and a companion paper is to present such an approach. The present model consists of a main space-time part and an internal part. In the former, approaches based upon conventional relativistic quantum mechanics are abandoned. Instead, quark and antiquark spinor equations are multiplied together and the product wave functions subsequently generalized to meson wave functions. The quark and the antiquark are assumed to interact via a strong massless pseudoscalar potential. The multiplication and generalization method is also applied to the associated potential equations. In the rest frame, the equations so obtained can be separated into a set of singlet and a set of triplet equations. These can be reduced to sets of three-dimensional equations in which a linear type of confinement arises without approximation. For pseudoscalar and vector mesons, the three-dimensional equations further reduce to one-dimensional nonlinear integrodifferential equations. An approximative analytical solution to these equations is obtained. This part is treated in Sections 2–4 and 6–8.

The second part of this paper deals with internal aspects of quarks and mesons. The composite quark–antiquark nature of mesons is emphasized at the expense of conventional unitary symmetry group classification. Internal space is introduced as well as mass operators operating on functions in this space. The approach draws upon analogy to that of the space-time part and puts space-time and internal coordinates on an equal footing. A simplified internal function and mass operator model is presented. This part is treated in Sections 5 and 9.

Application to data and discussions are given in Section 10.

2. BASIC IDEA

Conventional relativistic quantum mechanics and quantum field theory, from which the BS equation can be derived, and Pauli's spin-statistics theorem have been constructed for observable particles and fields. Therefore, they do not have to hold in their entirety for quarks and QCD gauge fields, which have not been observed. This is in contrast to the "standard" electroweak gauge model, which describes observed leptons and gauge quanta. Therefore, quarks in hadrons do not have to obey Pauli's theorem, so that "color" in QCD, introduced at first to appease this theorem, is not necessary. The role of color may be taken over by the internal coordinates to be introduced later.

Similarly, the BS equation does not have to hold for quarks in hadrons. This is made plausible by considering a ground-state meson whose BS wave function has 16 components, far exceeding the minimum of 4 wave function components (1 for singlet and 3 for triplet) needed to specify the orientations of the meson externally. The constituent particles of positronium, successfully accounted for by the BS equation, can in principle be separated and obey Dirac's equation. The quark and antiquark in a meson cannot be separated; the meson may therefore require a description different from that given by the BS equation.

In the absence of guidance from conventional field theory, the last paragraph provides a clue to the construction of hadron equations. Other hints are provided by the spinor form of the constituent quark or Dirac equations and by hadron data. The way in which these equations are constructed below is basically heuristic. Its choice is in principle free as long as the resulting equations are Lorentz covariant and can account for data, in the spirit of logical positivism.

3. HEURISTIC CONSTRUCTION OF A BS EQUATION

A clue to the construction of the meson equations below is provided by the following heuristic construction of a BS equation in the ladder approximation. Consider two interacting spin-1/2 particles. Particle A with mass m_{A} is acted upon by a massless pseudoscalar potential V_{PB} generated by particle B with mass m_{B} and vice versa. Each particle is described by a Dirac equation:

$$(i\gamma_{\mathrm{A}}^{\mu}\partial_{\mathrm{I}\mu}-m_{\mathrm{A}})\,\psi_{\mathrm{A}}(x_{\mathrm{I}})=-\gamma_{5\mathrm{A}}V_{\mathrm{PB}}(x_{\mathrm{I}})\,\psi_{\mathrm{A}}(x_{\mathrm{I}}) \tag{3.1}$$

$$(i\gamma_{\mathrm{B}}^{\nu}\partial_{\mathrm{II}\nu}-m_{\mathrm{B}})\,\psi_{\mathrm{B}}(x_{\mathrm{II}})=-\gamma_{5\mathrm{B}}V_{\mathrm{PA}}(x_{\mathrm{II}})\,\psi_{\mathrm{B}}(x_{\mathrm{II}}) \tag{3.2}$$

where x_{I} and x_{II} are space-time coordinates of particles A and B, respectively. Some of the symbols are defined in Appendix A. Multiply the left side of (3.1) by the left side of (3.2) and do the same to the right sides. The resulting equation is put in the form

$$\begin{aligned}(\gamma_{\mathrm{A}}^{\mu}\partial_{\mathrm{I}\mu}-m_{\mathrm{A}})(\gamma_{\mathrm{B}}^{\nu}\partial_{\mathrm{II}\nu}-m_{\mathrm{B}})\,\psi_{\mathrm{A}}(x_{\mathrm{I}})\,\psi_{\mathrm{B}}(x_{\mathrm{II}})\\ =V_{\mathrm{PB}}(x_{\mathrm{I}})\,V_{\mathrm{PA}}(x_{\mathrm{II}})\gamma_{5\mathrm{A}}\gamma_{5\mathrm{B}}\psi_{\mathrm{A}}(x_{\mathrm{I}})\,\psi_{\mathrm{B}}(x_{\mathrm{II}})\end{aligned}\tag{3.3}$$

Assume now, in an *ad hoc* manner, that the product functions in (3.3) separable in x_{I} and x_{II} can be generalized into nonseparable ones due to the mutual interaction of A and B:

$$\psi_{\mathrm{A}}(x_{\mathrm{I}})\,\psi_{\mathrm{B}}(x_{\mathrm{II}})\rightarrow\Psi(x_{\mathrm{I}},x_{\mathrm{II}})\tag{3.4}$$

$$V_{\mathrm{PB}}(x_{\mathrm{I}})\,V_{\mathrm{PA}}(x_{\mathrm{II}})\rightarrow\phi_{\mathrm{BS}}(|x_{\mathrm{I}}-x_{\mathrm{II}}|)\tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3), we obtain a BS equation in the ladder approximation. $\Psi(x_{\mathrm{I}}, x_{\mathrm{II}})$ is now a 16-component BS wave function amplitude and $\phi_{\mathrm{BS}}(|x_{\mathrm{I}}-x_{\mathrm{II}}|)$ some interaction function depending upon the specific form of the pseudoscalar interaction.

4. CONSTRUCTION OF SPINOR MESON EQUATIONS

As was mentioned in Section 2, the BS wave function has too many components for mesons. A clue here is given by a form of Weinberg's (1964) equations, which for second-rank symmetric spinors read

$$\begin{aligned}\partial_w^{a\dot{b}}\partial_w^{c\dot{d}}\chi_{\dot{b}\dot{d}}(x_w)&=-m_w^2\psi^{ac}(x_w)\\ \partial_{w\dot{d}a}\partial_{w\dot{c}c}\psi^{ac}(x_w)&=-m_w^2\chi_{\dot{b}\dot{d}}(x_w)\end{aligned}\tag{4.1}$$

Here, m_w is some constant mass and ∂_w and x_w are the same as those in Appendix A with the subscript $\mathrm{I}\rightarrow w$. The ψ and χ have six components, fewer than the ten components of the symmetric part of the BS wave function (3.4).

The quark and antiquark in a meson are assumed in this paper to act upon each other via a strong massless pseudoscalar interaction. This is supported by data which are consistent with a scalar confining potential in mesons (Lichtenberg, 1987). Equations (3.1) and (3.2) are now reinterpreted to represent a quark q_{A} and an antiquark $\bar{q}_{\mathrm{B}}$, respectively, under mutual massless pseudoscalar interaction. This is possible since (3.2) is covariant under the charge conjugation transformation

$$\psi_{\mathrm{B}}(x_{\mathrm{II}})\rightarrow i\gamma_{\mathrm{B}}^2\psi_{\mathrm{B}}^*(x_{\mathrm{II}})\tag{4.2}$$

which converts quark q_B to antiquark $\bar{q}_B$. The above reinterpretation is not to be taken literally, as it implies that quarks can be observed. Rather, (3.1) and (3.2) now serve as means in constructing the meson equations. The same holds for the following potential equations to be attached to (3.1) and (3.2):

$$\Box_{\mathrm{I}} V_{\mathrm{PB}}(x_{\mathrm{I}}) = g_{\mathrm{A}}\, g_{\mathrm{B}} \bar{\psi}_{\mathrm{B}}(x_{\mathrm{I}})\, \gamma_{5\mathrm{B}} \psi_{\mathrm{B}}(x_{\mathrm{I}}) \tag{4.3}$$

$$\Box_{\mathrm{II}} V_{\mathrm{PA}}(x_{\mathrm{II}}) = g_{\mathrm{B}}\, g_{\mathrm{A}} \bar{\psi}_{\mathrm{A}}(x_{\mathrm{II}})\, \gamma_{5\mathrm{A}} \psi_{\mathrm{A}}(x_{\mathrm{II}}) \tag{4.4}$$

where $g_A\, g_B$ is a coupling constant. Some symbols employed in this Section are also defined in Appendix A.

In spinor form, (3.1), (3.2), (4.3), and (4.4) are written as

$$\begin{aligned} \partial_{\mathrm{I}}^{a\dot{b}} \chi_{\mathrm{A}\dot{b}}(x_{\mathrm{I}}) - V_{\mathrm{PB}}(x_{\mathrm{I}})\, \psi_{\mathrm{A}}^{a}(x_{\mathrm{I}}) &= im_{\mathrm{A}} \psi_{\mathrm{A}}^{a}(x_{\mathrm{I}}) \\ \partial_{\mathrm{I}\dot{b}c} \psi_{\mathrm{A}}^{c}(x_{\mathrm{I}}) + V_{\mathrm{PB}}(x_{\mathrm{I}})\, \chi_{\mathrm{A}\dot{b}}(x_{\mathrm{I}}) &= im_{\mathrm{A}} \chi_{\mathrm{A}\dot{b}}(x_{\mathrm{I}}) \end{aligned} \tag{4.5}$$

$$\begin{aligned} \partial_{\mathrm{II}}^{d\dot{e}} \chi_{\mathrm{B}\dot{e}}(x_{\mathrm{II}}) - V_{\mathrm{PA}}(x_{\mathrm{II}})\, \psi_{\mathrm{B}}^{d}(x_{\mathrm{II}}) &= im_{\mathrm{B}}^{*} \psi_{\mathrm{B}}^{d}(x_{\mathrm{II}}) \\ \partial_{\mathrm{II}\dot{e}f} \psi_{\mathrm{B}}^{f}(x_{\mathrm{II}}) + V_{\mathrm{PA}}(x_{\mathrm{II}})\, \chi_{\mathrm{B}\dot{e}}(x_{\mathrm{II}}) &= im_{\mathrm{B}}^{*} \chi_{\mathrm{B}\dot{e}}(x_{\mathrm{II}}) \end{aligned} \tag{4.6}$$

$$\begin{aligned} \Box_{\mathrm{I}} V_{\mathrm{PB}}(x_{\mathrm{I}}) = i\tfrac{1}{2} g_{\mathrm{A}}\, g_{\mathrm{B}} (\psi_{\mathrm{B}}^{b}(x_{\mathrm{I}})\, \chi_{\mathrm{B}b}(x_{\mathrm{I}}) \\ - \chi_{\mathrm{B}\dot{b}}(x_{\mathrm{I}})\, \psi_{\mathrm{B}}^{\dot{b}}(x_{\mathrm{I}})) \end{aligned} \tag{4.7a}$$

$$\begin{aligned} \Box_{\mathrm{II}} V_{\mathrm{PA}}(x_{\mathrm{II}}) = i\tfrac{1}{2} g_{\mathrm{A}}\, g_{\mathrm{B}} (\psi_{\mathrm{A}}^{a}(x_{\mathrm{II}})\, \chi_{\mathrm{A}a}(x_{\mathrm{II}}) \\ - \chi_{\mathrm{A}\dot{a}}(x_{\mathrm{II}})\, \psi_{\mathrm{A}}^{\dot{a}}(x_{\mathrm{II}})) \end{aligned} \tag{4.7b}$$

where repeated space-time indices are summed and the dot implies complex conjugation. m_B here becomes m_B^* formally in view of (4.2).

Following Section 3, the left and right sides of the quark equations (4.5) are multiplied into the left and right sides, respectively, of the antiquark equations (4.6). Similarly, the left and right sides of (4.7a) are consistently multiplied into the corresponding sides of (4.7b). The resulting product wave functions are then generalized into nonseparable forms below, similar to (3.4) and (3.5):

$$\begin{aligned} \chi_{\mathrm{A}\dot{b}}(x_{\mathrm{I}})\, \psi_{\mathrm{B}}^{f}(x_{\mathrm{II}}) &\to \chi_{\dot{b}}^{f}(x_{\mathrm{I}}, x_{\mathrm{II}}) = \chi_{\dot{b}}^{f} \\ \psi_{\mathrm{A}}^{a}(x_{\mathrm{I}})\, \chi_{\mathrm{B}\dot{e}}(x_{\mathrm{II}}) &\to \psi_{\dot{e}}^{a}(x_{\mathrm{I}}, x_{\mathrm{II}}) = \psi_{\dot{e}}^{a} \\ \chi_{\mathrm{A}\dot{c}}(x_{\mathrm{I}})\, \chi_{\mathrm{B}\dot{e}}(x_{\mathrm{II}}) &\to \chi_{\dot{c}\dot{e}}(x_{\mathrm{I}}, x_{\mathrm{II}}) = \chi_{\dot{c}\dot{e}} \\ \psi_{\mathrm{A}}^{b}(x_{\mathrm{I}})\, \psi_{\mathrm{B}}^{f}(x_{\mathrm{II}}) &\to \psi^{bf}(x_{\mathrm{I}}, x_{\mathrm{II}}) = \psi^{bf} \end{aligned} \tag{4.8a}$$

$$V_{\mathrm{PB}}(x_{\mathrm{I}})\, V_{\mathrm{PA}}(x_{\mathrm{II}}) \to \phi_{\mathrm{P}}(x_{\mathrm{I}}, x_{\mathrm{II}}) = \phi_{\mathrm{P}} \tag{4.8b}$$

These relations, together with the corresponding generalization for internal functions and operator in (5.3) below, constitute the basic hypothesis of the present model and mark the departure from conventional relativistic quantum mechanics.

The equations so generalized read

$$\partial_{\mathrm{I}}^{a\dot{b}} \chi_{\dot{b}\dot{e}} \partial_{\mathrm{II}}^{\dot{e}d} - \phi_{\mathrm{P}} \psi^{ad} + V_{\mathrm{PA}} \partial_{\mathrm{I}}^{a\dot{b}} \chi_{\dot{b}}^{d} - V_{\mathrm{PB}} \psi_{\dot{e}}^{a} \partial_{\mathrm{II}}^{\dot{e}d} = -m_{\mathrm{A}} m_{\mathrm{B}}^{*} \psi^{ad}$$

$$\partial_{\mathrm{I}\dot{c}b} \psi^{bf} \partial_{\mathrm{II}f\dot{e}} - \phi_{\mathrm{P}} \chi_{\dot{c}\dot{e}} - V_{\mathrm{PA}} \partial_{\mathrm{I}\dot{c}b} \psi_{\dot{e}}^{b} + V_{\mathrm{PB}} \chi_{\dot{c}}^{f} \partial_{\mathrm{II}f\dot{e}} = -m_{\mathrm{A}} m_{\mathrm{B}}^{*} \chi_{\dot{c}\dot{e}} \qquad (4.9a)$$

$$\partial_{\mathrm{I}}^{a\dot{b}} \chi_{\dot{b}}^{f} \partial_{\mathrm{II}f\dot{e}} - \phi_{\mathrm{P}} \psi_{\dot{e}}^{a} + V_{\mathrm{PA}} \partial_{\mathrm{I}}^{a\dot{b}} \chi_{\dot{b}\dot{e}} - V_{\mathrm{PB}} \psi^{af} \partial_{\mathrm{II}f\dot{e}} = -m_{\mathrm{A}} m_{\mathrm{B}}^{*} \psi_{\dot{e}}^{a}$$

$$\partial_{\mathrm{I}\dot{c}b} \psi_{\dot{e}}^{b} \partial_{\mathrm{II}}^{\dot{e}d} - \phi_{\mathrm{P}} \chi_{\dot{c}}^{d} - V_{\mathrm{PA}} \partial_{\mathrm{I}\dot{c}b} \psi^{bd} + V_{\mathrm{PB}} \chi_{\dot{c}\dot{e}} \partial_{\mathrm{II}}^{\dot{e}d} = -m_{\mathrm{A}} m_{\mathrm{B}}^{*} \chi_{\dot{c}}^{d} \qquad (4.9b)$$

$$\begin{aligned} \Box_{\mathrm{I}} \Box_{\mathrm{II}} \phi_{\mathrm{P}}(x_{\mathrm{I}}, x_{\mathrm{II}}) = \tfrac{1}{4} g_{\mathrm{A}}^{2} g_{\mathrm{B}}^{2} [& \psi_{\dot{b}}^{a}(x_{\mathrm{II}}, x_{\mathrm{I}}) (\chi_{\dot{a}}^{b}(x_{\mathrm{II}}, x_{\mathrm{I}}))^{*} \\ & + \chi_{\dot{a}}^{b}(x_{\mathrm{II}}, x_{\mathrm{I}}) (\psi_{\dot{b}}^{a}(x_{\mathrm{II}}, x_{\mathrm{I}}))^{*} - \psi^{ab}(x_{\mathrm{II}}, x_{\mathrm{I}}) (\chi_{\dot{a}\dot{b}}(x_{\mathrm{II}}, x_{\mathrm{I}}))^{*} \\ & - \chi_{\dot{a}\dot{b}}(x_{\mathrm{II}}, x_{\mathrm{I}}) (\psi^{ab}(x_{\mathrm{II}}, x_{\mathrm{I}}))^{*}] \end{aligned} \qquad (4.10)$$

where $\chi\partial$ means $\partial\chi$. Equations (4.5)–(4.7), (4.9), and (4.10) constitute a self-consistent set of equations describing, tentatively, a quark $q_{\mathrm{A}}(\psi_{\mathrm{A}}, \chi_{\mathrm{A}})$, an antiquark $\bar{q}_{\mathrm{B}}(\psi_{\mathrm{B}}, \chi_{\mathrm{B}})$, a rank-2 tensor field (ψ^{ad}, $\chi_{\dot{c}\dot{e}}$), and a mixed rank-2 tensor field ($\psi_{\dot{e}}^{a}$, $\chi_{\dot{c}}^{d}$) interacting via V_{PA}, V_{PB}, and ϕ_{P}. In the absence of interactions, (4.9a) reduces to the Weinberg equations (4.1) in the limit of $x_{\mathrm{I}} \to x_{\mathrm{II}} = x_{w}$ with $m_{\mathrm{A}} \to m_{\mathrm{B}}^{*} = m_{w}$.

Consider now a possible representation of mesons by this set of equations. The mesons are obviously represented by the mixed tensor fields $\psi_{\dot{e}}^{a}$ and $\chi_{\dot{c}}^{d}$ in (4.9b) and (4.10). Since no free quark exists, the quark and antiquark fields of (4.5) and (4.6) can be put to zero. From (4.7) it follows that also $V_{\mathrm{PA}} = V_{\mathrm{PB}} = 0$, in agreement with absence of massless pseudo scalar particles. Due to the covariance of quark and antiquark wave equations under (4.2), the antiquark equations may also be reinterpreted as quark equations. In this case, ψ^{ad} and $\chi_{\dot{c}\dot{e}}$ of (4.9a) are associated with what may be called a diquark. This arbitrariness in interpretation and apparent absence of free diquark suggest that ψ^{ad} and $\chi_{\dot{c}\dot{e}}$ can likewise be dropped. With these interpretations, (4.9a) and the last two terms in the brackets of (4.10) also drop out and (4.9b) and (4.10) become

$$\partial_{\mathrm{I}}^{a\dot{b}} \chi_{\dot{b}}^{f}(x_{\mathrm{I}}, x_{\mathrm{II}}) \partial_{\mathrm{II}f\dot{e}} = (\phi_{\mathrm{P}}(x_{\mathrm{I}}, x_{\mathrm{II}}) - M_{m}^{2}) \psi_{\dot{e}}^{a}(x_{\mathrm{I}}, x_{\mathrm{II}}) \qquad (4.11a)$$

$$\partial_{\mathrm{I}\dot{c}b} \psi_{\dot{e}}^{b}(x_{\mathrm{I}}, x_{\mathrm{II}}) \partial_{\mathrm{II}}^{\dot{e}d} = (\phi_{\mathrm{P}}(x_{\mathrm{I}}, x_{\mathrm{II}}) - M_{m}^{2}) \chi_{\dot{c}}^{d}(x_{\mathrm{I}}, x_{\mathrm{II}}) \qquad (4.11b)$$

$$\Box_{\mathrm{I}} \Box_{\mathrm{II}} \phi_{\mathrm{P}}(x_{\mathrm{I}}, x_{\mathrm{II}}) = \tfrac{1}{2} g_{\mathrm{A}}^{2} g_{\mathrm{B}}^{2} \operatorname{Re}[\psi_{\dot{b}}^{a}(x_{\mathrm{II}}, x_{\mathrm{I}}) (\chi_{\dot{a}}^{b}(x_{\mathrm{II}}, x_{\mathrm{I}}))^{*}] \qquad (4.12)$$

where Re denotes real part and $m_{\mathrm{A}} m_{\mathrm{B}}^{*}$ has been replaced by M_{m}^{2} according to the assignment below (5.4). These are the space-time equations assigned to mesons. Lorentz covariance is manifest from their spinor form. Under spatial reflection, (4.11a) and (4.11b) turn into each other, (4.12) is invariant, and $\chi^a_{\dot{e}}(x^0_{\rm I}, \mathbf{x}_{\rm I}, x^0_{\rm II}, \mathbf{x}_{\rm II}) \to \psi^a_{\dot{e}}(x^0_{\rm I}, -\mathbf{x}_{\rm I}, x^0_{\rm II}, -\mathbf{x}_{\rm II})$ together with the same relation with $\chi \leftrightarrow \psi$.

The self-consistent nature of these equations is similar to that of the usual basic equations for pion–nucleon scattering; the difference lies in the order of the equations.

5. INCLUSION OF INTERNAL COORDINATES

Hadrons are also classified by internal properties, like up, down, etc. Let $z^1, z^2, \ldots, z^n$ be a set of complex variables providing a space in which unitary transformations U_n can be carried out (e.g., Bég and Ruegg *et al.*, 1965; Sharp and von Baeyer, 1965). Let the up (u), down (d), strange (s), charm (c), and bottom (b) flavored quarks be represented by $\xi^1(z)$, $\xi^2(z)$, $\xi^4(z)$, and $\xi^5(z)$, respectively, transforming like $z^1, \ldots, z^5$ under U_n. Let $(z^\nu)^* = z_\nu$ and $(\xi^\nu(z))^* = \xi_\nu(z)$ be associated with the corresponding antiquarks. Here, z denotes z^ν and z_ν with $\nu = 1, 2, \ldots, n$.

Equations (4.5) and (4.6) are now generalized to include internal properties; (4.5) is multiplied by $\xi^p_{\rm A}(z_{\rm I})$ from the right with m_A therein replaced by an internal operator $m_{{\rm A}op}(z_{\rm I}, \partial/\partial z_{\rm I})$ and (4.6) is multiplied by $\xi_{{\rm B}r}(z_{\rm II})$ with $m^*_{\rm B}$ replaced by $m_{{\rm B}op}(z_{\rm II}, \partial/\partial z_{\rm II})$. These now become

$$\partial^{a\dot{b}}_{\rm I} \chi_{{\rm A}\dot{b}}(x_{\rm I})\, \xi^p_{\rm A}(z_{\rm I}) - V_{\rm PB}(x_{\rm I})\, \psi^a_{\rm A}(x_{\rm I})\, \xi^p_{\rm A}(z_{\rm I})$$
$$= im_{{\rm A}op}(z_{\rm I}, \partial/\partial z_{\rm I})\, \psi^a_{\rm A}(x_{\rm I})\, \xi^p_{\rm A}(z_{\rm I}) \tag{5.1a}$$

$$\partial_{{\rm I}\dot{b}c} \psi^c_{\rm A}(x_{\rm I})\, \xi^p_{\rm A}(z_{\rm I}) + V_{\rm PB}(x_{\rm I})\, \chi_{{\rm A}\dot{b}}(x_{\rm I})\, \xi^p_{\rm A}(z_{\rm I})$$
$$= im_{{\rm A}op}(z_{\rm I}, \partial/\partial z_{\rm I})\, \chi_{{\rm A}\dot{b}}(x_{\rm I})\, \xi^p_{\rm A}(z_{\rm I}) \tag{5.1b}$$

$$\partial^{d\dot{e}}_{\rm II} \chi_{{\rm B}\dot{e}}(x_{\rm II})\, \xi_{{\rm B}r}(z_{\rm II}) - V_{\rm PA}(x_{\rm II})\, \psi^d_{\rm B}(x_{\rm II})\, \xi_{{\rm B}r}(z_{\rm II})$$
$$= im^*_{{\rm B}op}/z_{\rm II}, \partial/\partial z_{\rm II})\, \psi^d_{\rm B}(x_{\rm II})\, \xi_{{\rm B}r}(z_{\rm II}) \tag{5.2a}$$

$$\partial_{{\rm II}\dot{e}f} \psi^f_{\rm B}(x_{\rm II})\, \xi_{{\rm B}r}(z_{\rm II}) + V_{\rm PA}(x_{\rm II})\, \chi_{{\rm B}\dot{e}}(x_{\rm II})\, \xi_{{\rm B}r}(z_{\rm II})$$
$$= im^*_{{\rm B}op}(z_{\rm II}, \partial/\partial z_{\rm II})\, \chi_{{\rm B}\dot{e}}(x_{\rm II})\, \xi_{{\rm B}r}(z_{\rm II}) \tag{5.2b}$$

$\psi^a_{\rm A}(x_{\rm I})\, \xi^p_{\rm A}(z_{\rm I})$ and $\chi_{{\rm A}\dot{b}}(x_{\rm I})\, \xi^p_{\rm A}(z_{\rm I})$ in (5.1) now represent the total quark wave functions. Equations (5.1) and (5.2) are multiplied together and the generalization (4.8) is repeated. In the resulting equations which replace (4.9), the generalizations

$$\xi^p_{\rm A}(z_{\rm I})\, \xi_{{\rm B}r}(z_{\rm II}) \to \xi^p{}_r(z_{\rm I}, z_{\rm II}) \tag{5.3a}$$

$$m_{{\rm A}op}(z_{\rm I}, \partial/\partial z_{\rm I})\, m^*_{{\rm B}op}(z_{\rm II}, \partial/\partial z_{\rm II}) \to m_{2op}(z_{\rm I}, \partial/\partial z_{\rm I}, z_{\rm II}, \partial/\partial z_{\rm II}) = m_{2op} \tag{5.3b}$$

are made by formal analogy with (4.8a) and (4.8b), respectively. Analogous to the right sides of (4.8), the right sides of (5.3) are generally not separable in $z_{\rm I}$ and $z_{\rm II}$. Repeating the reasoning following (4.10), (4.11) is replaced by

$$\begin{aligned} &\partial_{\rm I}^{a\dot{b}}\chi_{\dot{b}}^{f}(x_{\rm I}, x_{\rm II})\,\xi^{p}{}_{r}(z_{\rm I}, z_{\rm II})\,\partial_{{\rm II}f\dot{e}} \\ &\qquad = (\phi_{\rm P}(x_{\rm I}, x_{\rm II}) - m_{2op})\,\psi_{\dot{e}}^{a}(x_{\rm I}, x_{\rm II})\,\xi^{p}{}_{r}(z_{\rm I}, z_{\rm II}) \end{aligned} \tag{5.4a}$$

$$\begin{aligned} &\partial_{{\rm I}\dot{c}b}\psi_{\dot{e}}^{b}(x_{\rm I}, x_{\rm II})\,\xi^{p}{}_{r}(z_{\rm I}, z_{\rm II})\,\partial_{\rm II}^{\dot{e}d} \\ &\qquad = (\phi_{\rm P}(x_{\rm I}, x_{\rm II}) - m_{2op})\,\chi_{\dot{c}}^{d}(x_{\rm I}, x_{\rm II})\,\xi^{p}{}_{r}(z_{\rm I}, z_{\rm II}) \end{aligned} \tag{5.4b}$$

The total meson functions are then $\chi_{\dot{b}}^{f}(x_{\rm I}, x_{\rm II})\,\xi^{p}{}_{r}(z_{\rm I}, z_{\rm II})$ and $\psi_{\dot{e}}^{b}(x_{\rm I}, x_{\rm II})\,\xi^{p}{}_{r}(z_{\rm I}, z_{\rm II})$, which have to be eigenfunctions of m_{2op} having an eigenvalue M_m^2. Equation (5.4) can be separated into a space-time part (4.11) and an internal part

$$m_{2op}(z_{\rm I}, \partial/\partial z_{\rm I}, z_{\rm II}, \partial/\partial z_{\rm II})\,\xi^{p}{}_{r}(z_{\rm I}, z_{\rm II}) = M_m^2\xi^{p}{}_{r}(z_{\rm I}, z_{\rm II}) \tag{5.5}$$

Equations (5.4), (5.5), and (4.12) are the proposed spinor meson equations in the present model subjected to the symmetry condition (9.2) below.

6. REDUCTION OF THE SPACE-TIME EQUATIONS

Consider first (4.11) and (4.12) and put

$$\begin{aligned} \psi_{\dot{a}}^{b}(x_{\rm I}, x_{\rm II}) &= \psi^{b\dot{c}}(x_{\rm I}, x_{\rm II})\varepsilon_{\dot{c}\dot{e}} \\ \chi_{\dot{c}}^{d}(x_{\rm I}, x_{\rm II}) &= \chi_{\dot{c}e}(x_{\rm I}, x_{\rm II})\varepsilon^{ed} \\ \psi^{b\dot{c}} &= \psi_0'(x_{\rm I}, x_{\rm II}) - \boldsymbol{\sigma}^{b\dot{c}}\boldsymbol{\psi}'(x_{\rm I}, x_{\rm II}) \\ \chi_{\dot{c}e} &= \chi_0'(x_{\rm I}, x_{\rm II}) + \boldsymbol{\sigma}_{\dot{c}e}\boldsymbol{\chi}'(x_{\rm I}, x_{\rm II}) \end{aligned} \tag{6.1a}$$

Here ε^{ed} and $\varepsilon_{\dot{c}\dot{e}}$ are the usual antisymmetric index raising and lowering operators and $\boldsymbol{\sigma}$ are the Pauli matrices, so that

$$\begin{aligned} \psi_0' &= \tfrac{1}{2}(\psi^{1\dot{1}} + \psi^{2\dot{2}}) = \tfrac{1}{2}(\psi_{\dot{2}}^{1} - \psi_{\dot{1}}^{2}) \\ \boldsymbol{\psi}' &= \tfrac{1}{2}(\psi_{\dot{2}}^{2} - \psi_{\dot{1}}^{1},\ i(\psi_{\dot{2}}^{2} + \psi_{\dot{1}}^{1}),\ \psi_{\dot{2}}^{1} + \psi_{\dot{1}}^{2}) \end{aligned} \tag{6.1b}$$

Thus, the singlet ψ_0' is the antisymmetric part $\psi_{\dot{b}}^{a} = -\psi_{\dot{a}}^{b}$, while $\boldsymbol{\psi}'$ is the symmetric part $\psi_{\dot{b}}^{a} = \psi_{\dot{a}}^{b}$. The same also holds for χ_0' and $\boldsymbol{\chi}'$.

Introduce the relative and laboratory conditions

$$x^\mu = x_{\rm II}^\mu - x_{\rm I}^\mu, \qquad X^\mu = (1 - a_m)x_{\rm I}^\mu + a_m x_{\rm II}^\mu \tag{6.2}$$

where a_m is an arbitrary constant. Consider a free meson and let

$$\psi_0'(x_{\mathrm{I}}, x_{\mathrm{II}}) = e^{-iK_\mu X^\mu}\psi_0'(x^\mu), \qquad \boldsymbol{\psi}'(x_{\mathrm{I}}, x_{\mathrm{II}}) = e^{-iK_\mu X^\mu}\boldsymbol{\psi}'(x^\mu) \tag{6.3}$$

together with analogous expressions for χ_0' and $\boldsymbol{\chi}'$. Here, $K_\mu = (E_0, -\mathbf{K})$, E_0 denotes the rest mass of the meson, and $\mathbf{K}$ its momentum. With (6.1) and (6.2), (4.11a) is written in the form

$$\begin{aligned}
&[a_m(1-a_m)(E_0^2+\mathbf{K}^2)+\partial_0^2+\Delta+i(1-2a_m)(E_0\partial_0-\mathbf{K}\partial)\chi_0' \\
&\quad+(2\partial_0\partial+i(1-2a_m)(E_0\partial-\mathbf{K}\partial_0)-2a_m(1-a_m)\,E_0\mathbf{K}+\mathbf{K}\times\partial]\,\boldsymbol{\sigma}\chi_0' \\
&\quad+[a_m(1-a_m)(E_0^2-\mathbf{K}^2)+\partial_0^2-\Delta+i(1-2a_m)(E_0\partial_0+\mathbf{K}\partial)]\,\boldsymbol{\sigma}\boldsymbol{\chi}' \\
&\quad+E_0\boldsymbol{\sigma}(\partial x\boldsymbol{\chi}')+[2\boldsymbol{\sigma}\partial+2\partial_0+i(1-2a_m)(E_0-\boldsymbol{\sigma}\mathbf{K})]\,\partial\boldsymbol{\chi}' \\
&\quad-[i(1-2a_m)(\partial_0+\boldsymbol{\sigma}\partial)+2a_m(1-a_m)(E_0-\boldsymbol{\sigma}\mathbf{K})]\,\mathbf{K}\boldsymbol{\chi}' \\
&\quad+(\partial+\partial_0\boldsymbol{\sigma})(\mathbf{K}\times\boldsymbol{\chi}')=(\phi_{\mathrm{P}}-M_m^2)(\psi_0'-\boldsymbol{\sigma}\boldsymbol{\psi}')
\end{aligned} \tag{6.4}$$

where $\partial_0 = \partial/\partial x^0$, $\partial = \partial/\partial \mathbf{x}$ and $\Delta = \partial\partial$. Equation (4.11b) becomes the same equation with $\chi \leftrightarrow \psi$ and opposite signs for the cross products. Solutions having the following relative time x^0 dependence are sought

$$\varphi(x^\mu) = e^{i\omega_0 x^0}\varphi(\mathbf{x}), \qquad \varphi = \psi_0', \boldsymbol{\psi}', \chi_0', \boldsymbol{\chi}' \tag{6.5}$$

where ω_0 is the relative energy among the quarks. The arbitrary constant a_m is chosen to be

$$a_m = \omega_0/E_0 + \tfrac{1}{2} \tag{6.6}$$

The time dependence of $\psi_{\dot{e}}^a$ and χ_b^d then takes the form of $\exp[-iE_0(x_{\mathrm{I}}^0 + x_{\mathrm{II}}^0)/2]$. The rest frame $\mathbf{K}=0$ will be worked out below. The singlet and triplet parts of (6.4) can now be decoupled. Equation (4.11b) is treated in the same way. The set of decoupled equations reads

$$(E_0^2/4+\Delta)\,\chi_0'(\mathbf{x}) = (\phi_{\mathrm{P}}(\mathbf{x}) - M_m^2)\,\psi_0'(\mathbf{x}) \tag{6.7a}$$

$$(E_0^2/4+\Delta)\,\psi_0'(\mathbf{x}) = (\phi_{\mathrm{P}}(\mathbf{x}) - M_m^2)\,\chi_0'(\mathbf{x}) \tag{6.7b}$$

$$(-E_0^2/4+\Delta)\,\boldsymbol{\chi}'(\mathbf{x}) - 2\partial(\partial\boldsymbol{\chi}'(\mathbf{x})) - E_0\,\partial\times\boldsymbol{\chi}'(\mathbf{x}) = (\phi_{\mathrm{P}}(\mathbf{x}) - M_m^2)\,\boldsymbol{\psi}'(\mathbf{x}) \tag{6.8a}$$

$$(-E_0^2/4+\Delta)\,\boldsymbol{\psi}'(\mathbf{x}) - 2\partial(\partial\boldsymbol{\psi}'(\mathbf{x})) + E_0\,\partial\times\boldsymbol{\psi}'(\mathbf{x}) = (\phi_{\mathrm{P}}(\mathbf{x}) - M_m^2)\,\boldsymbol{\chi}'(\mathbf{x}) \tag{6.8b}$$

Here $\phi_{\mathrm{P}}(\mathbf{x})$ is found by combining (4.12) with (6.1)–(6.3) and (6.5);

$$\Delta\Delta\phi_{\mathrm{P}}(\mathbf{x}) = g_{\mathrm{A}}^2\, g_{\mathrm{B}}^2\, \mathrm{Re}(\boldsymbol{\psi}'(-\mathbf{x})\,\boldsymbol{\chi}'^*(-\mathbf{x}) - \psi_0'(-\mathbf{x})\,\chi_0'^*(-\mathbf{x})) \tag{6.9}$$

which also holds for $\mathbf{K} \neq 0$ and without using (6.6).

Let S denote the spin of the ground-state mesons; then $S=0$ and 1 refer to the singlet functions ψ_0' and χ_0' and the triplet functions $\boldsymbol{\psi}'$ and $\boldsymbol{\chi}'$, respectively. The lowest order equation obtainable from (6.7) is found by putting $\psi_0'(\mathbf{x}) = \pm\chi_0'(\mathbf{x})$. Under spatial reflection, (6.2) and a statement below (4.12) show that the $S=0$ meson wave functions transform as

$$[\psi_0'(\mathbf{x}), \chi_0'(\mathbf{x})] = [\psi_0'(\mathbf{x}), \pm\psi_0'(\mathbf{x})]$$
$$\to [\chi_0'(-\mathbf{x}), \psi_0'(-\mathbf{x})] = \pm[\psi_0'(-\mathbf{x}), \pm\psi_0'(-\mathbf{x})]$$

In the absence of angular excitation, $\psi_0'(\mathbf{x}) = \psi_0'(|\mathbf{x}|)$, the $+$ and $-$ signs then refer to J^P (J = total spin, P = parity) $= 0^+$ and 0^- or to scalar and pseudoscalar mesons, respectively. For 0^+, (6.7) leads to a sign change of ϕ_{PO} in (7.3) below and nonconfinement. For 0^-, (6.7) becomes

$$(E_0^2/4 - M_m^2 + \Delta + \phi_{\mathrm{P}}(\mathbf{x}))\,\psi_0(\mathbf{x}) = 0 \tag{6.10}$$

$$\psi_0(\mathbf{x}) = \psi_0'(\mathbf{x})\, g_{\mathrm{A}}\, g_{\mathrm{B}} \tag{6.11}$$

which together with (6.9), putting $\boldsymbol{\psi}'$ and $\boldsymbol{\chi}'$ to zero, form a set of nonlinear equations describing $S=0$ mesons at rest. Analogously, (6.9) without the $\psi_0'\chi_0'^*$ term and (6.8) describe $S=1$ mesons at rest.

Equation (6.10) shows that E_0^2, the square of the meson mass, consists of a bare quark mass part M_m^2, a quark-antiquark interaction part ϕ_{P} and a kinetic energy part Δ associated with the relative motion of the quark and antiquark.

For slow mesons or small $\mathbf{K}$, (6.4) shows that $\boldsymbol{\psi}'$, $\boldsymbol{\chi}'$ and ψ_0', χ_0' will be of order $|\mathbf{K}|$ for $S=0$ and 1, respectively. Thus the triplet and singlet wave functions corresponds to the small and large components for $S=0$ and vice versa for $S=1$, analogous to those for $S=1/2$ particles described by Dirac's free particle equation.

For very fast mesons, $\mathbf{K} = (0, 0, K\to\infty)$; ϕ_{P}, M_m^2 and the ∂ terms in (6.4) can be neglected and the required relation $E_0^2 = K^2$ is satisfied if

$$\omega_0 = 0, \qquad \chi_3' = \chi_0' \tag{6.12a}$$

$$\omega_0 = \pm K, \qquad \chi_3' = -\chi_0', \qquad \chi_1' = \pm i\chi_2' \tag{6.12b}$$

where $\boldsymbol{\chi}' = (\chi_1', \chi_2', \chi_3')$. The same result also holds for $\chi \to \psi$.

7. LINEAR CONFINEMENT AND THE $S=0$ EQUATIONS

The Green's function for (6.9) satisfies

$$\Delta\Delta G_m(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}') \tag{7.1a}$$

$$G_m(\mathbf{x}, \mathbf{x}') = -\frac{1}{8\pi}|\mathbf{x} - \mathbf{x}'| \tag{7.1b}$$

There are, however, also homogeneous solutions to (6.9) proportional to $|\mathbf{x}|^2$, $|\mathbf{x}|^{-1}$, and a constant. $|\mathbf{x}_{\mathrm{I}}|^{-1}$ and $|\mathbf{x}_{\mathrm{II}}|^{-1}$ are also proportional to the Green's function associated with the massless pseudoscalar interaction of (4.7) if the quark wave functions ψ and χ have the time dependence of the type of (6.5). In this case, (4.7) also allows a constant as a homogeneous solution. Now this time dependence cannot be assumed in general. In such cases, the right sides of (4.7) will be time dependent and the constant as a homogeneous solution to (4.7) must vanish, apart from a modification of the Green's function itself. The same holds if the interaction is assumed to be slightly massive. Since the constant homogeneous solutions to (6.9) and (4.7) are of the same nature, the former can also put to zero here, in view that the present equations may be some idealization of more general ones. It can be shown that the homogeneous solution $|\mathbf{x}|^2$ to (6.9) is also a counterpart to the constant homogeneous solution to (4.7) and can therefore likewise be put to zero.

With these considerations, (6.9), (6.11), and (7.1) for the $S=0$ mesons can be combined to yield

$$\phi_{\mathrm{P}}(\mathbf{x}) = -\frac{1}{8\pi}\int d^3\mathbf{x}'\,|\mathbf{x}-\mathbf{x}'|\,|\psi_0(-\mathbf{x}')|^2 + d_{m0}/|\mathbf{x}| \tag{7.2}$$

where d_{m0} is a constant. The nonlinear nature of (6.7)–(6.9) makes it possible for the coupling constant $g_{\mathrm{A}}\,g_{\mathrm{B}}$ to be absorbed into ψ_0 as in (6.11). At large separations, (7.2) shows that $\phi_{\mathrm{P}}(\mathbf{x}) \propto -|\mathbf{x}|$ is linearly confining. Near $\mathbf{x}=0$, the $|\mathbf{x}|^{-1}$ term dominates. Apart from its coupling to the $S=0$ equation (6.10) via ψ_0, (7.2) is just the type of potential successfully employed in potential models for mesons (de Rújula *et al.*, 1975; Lichtenberg, 1987).

The linear confinement arises naturally from the formalism and cannot be interpreted in conventional terms, as the construction of the theory departs from conventional methods. Such a linear potential is known in the literature in another context. Further, $\psi_0(\mathbf{x})$ is fixed by the nonlinear equations (6.10) and (7.2) and its interpretation is also unknown. Should the theory be useful, new concepts may then be formed to join existing ones to account for the mathematical formalism and data in words.

Equations (6.10) and (7.2) are the space-time part of the $S=0$ meson equations and form a single nonlinear integrodifferential equation in three dimensions. Let now $\mathbf{x}$ be converted to spherical coordinates (r, ϑ, φ). In the absence of angular excitation, $\psi_0(\mathbf{x}) \to \psi_0(r)$ and $\phi_P(\mathbf{x}) \to \phi_P(r)$. The three-dimensional equations are then reduced to the following equations equivalent to a nonlinear ordinary singular integrodifferential equation:

$$[E_{00}^2/4 - M_m^2 + \partial^2/\partial r^2 + 2\partial/\partial r + \phi_{P0}(r)]\,\psi_0(r) = 0 \tag{7.3}$$

$$\phi_{P0} = -\tfrac{1}{6}\left[\int_0^r dr'\, \psi_0^2(r')\, r'^2(3r + r'^2/r) + \int_r^\infty dr'\, \psi_0^2(r')\, r'(3r'^2 + r^2)\right] + d_{m0}/r \tag{7.4}$$

where E_{00} replaces E_0 associated with $S=0$. These equations are assigned to pseudoscalar mesons, as is shown in the derivation of (6.10).

In the $r \to 0$ and ∞ limits,

$$\psi_0(r \to 0) = d_0 \qquad \text{or} \qquad d_0' r^{-1} \tag{7.5a}$$

$$\phi_{P0}(r \to 0) = -\tfrac{1}{2}\int_0^\infty r^3 \psi_0^2(r)\, dr + d_{m0}/r \tag{7.5b}$$

$$\psi_0(r \to \infty) = d_{0\infty}(\beta_{m0}^{1/3} r)^{1/2}\, K_{1/3}(2(\beta_{m0}^{1/3} r)^{3/2}/3) \tag{7.6a}$$

$$\phi_{P0}(r \to \infty) = -\tfrac{1}{2} r \int_0^\infty r'^2 \psi_0^2(r')\, dr' = -\beta_{m0} r \tag{7.6b}$$

where K denotes the modified Bessel function of the second kind and the d's are nonfree constants fixed by the nonlinear equations. Should eigensolutions exist, (7.3)–(7.5) can be solved numerically on a computer.

An estimate of the lower limit of the eigenvalue of (7.3)–(7.5) can be obtained if it is assumed that the solution $\psi_0(r)$ is regular at $r=0$ and is largely confined to that region, so that the integrals in (7.4) can be neglected. A hydrogen atom type of spectrum is then found:

$$E_{00}^2/4 - M_m^2 = -(d_{m0}/2(n_{r0}+1))^2 \tag{7.7}$$

where n_{r0} denotes the radial quantum number.

The more general (6.10) and (7.2) are not separable, due to the angular dependence of $\phi_P(\mathbf{x})$. However, this angular dependence vanishes at the $r \to 0$ and ∞ limits, where $\psi_0(\mathbf{x})$ can be separated,

$$\psi_0(\mathbf{x} \to 0) \approx d_{0l} r^l Y_{lm}(\vartheta, \varphi) \qquad \text{or} \qquad d_{0l}' r^{-l-1} Y_{lm}(\vartheta, \varphi) \tag{7.8}$$

$$\psi_0(\mathbf{x} \to \infty) = d_{0\infty l}(\beta_{m0l} r)^{1/2}\, K_{1/3}(2(\beta_{m0l} r)^{1/2}/3) \tag{7.9}$$

Here, the d's are constants, Y_{lm} is the usual spherical harmonics, and β_{m0l} is given by (7.11b) below. In these limits, (6.10) becomes

$$[E_0^2/4 - M_m^2 + \partial^2/\partial r^2 + 2\partial/\partial r - l(l+1)/r^2 + \phi_P(\mathbf{x} \to {}^{0}_{\infty})]\, \psi_0(\mathbf{x} \to {}^{0}_{\infty}) = 0 \tag{7.10}$$

with

$$\phi_P(\mathbf{x} \to 0) = -\frac{1}{8\pi} \int d^3\mathbf{x}\, |\mathbf{x}|\, |\psi_0(-\mathbf{x})|^2 + d_{m0}/r \tag{7.11a}$$

$$\phi_P(\mathbf{x} \to \infty) \approx -\frac{1}{8\pi} r \int d^3\mathbf{x}\, |\psi_0(-\mathbf{x})|^2 = -\beta_{m0} r \tag{7.11b}$$

assuming that a confined solution $\psi_0(\mathbf{x})$ exists.

8. $S=1$ EQUATIONS

For the $S=1$ mesons, the counterpart to (7.2) is

$$\phi_P(\mathbf{x}) = -\frac{1}{8\pi} \int d^3\mathbf{x}'\, |\mathbf{x}-\mathbf{x}'|\, \mathrm{Re}(\psi(-\mathbf{x}')\, \chi^*(-\mathbf{x}')) + d_{m1}/r \tag{8.1a}$$

$$\psi(\mathbf{x}) = \psi'(\mathbf{x})\, g_A\, g_B, \qquad \chi(\mathbf{x}) = \chi'(\mathbf{x})\, g_A\, g_B \tag{8.1b}$$

where d_{m1} is a constant. Equations (8.1) and (6.8) are the equations for $S=1$ mesons. This set of three-dimensional equations is likewise not separable generally. In the absence of angular excitation, however, the ansatz

$$\psi(\mathbf{x}) = \mathbf{x}\psi_1(r)/r = \pm\chi(\mathbf{x}) \tag{8.2}$$

reduces the pair of three-dimensional equations (6.8) into two one-dimensional equations. Under spatial reflection, the $S=1$ wave functions transform like

$$[\psi(\mathbf{x}), \chi(\mathbf{x})] = [\psi(\mathbf{x}), \pm\psi(\mathbf{x})] \to [\chi(-\mathbf{x}), \psi(-\mathbf{x})] = \mp[\psi(\mathbf{x}), \pm\psi(\mathbf{x})]$$

The + and − signs refer to $J^P = 1^-$ and 1^+, respectively, when (8.2) holds. Analogous to the 0^+ case preceding (6.10), the 1^+ equation leads to nonconfinement and is therefore dropped. The 1^- form of (6.8), (8.1), and (8.2) can be worked out to yield

$$[E_{01}^2/4 - M_m^2 + \partial^2/\partial r^2 + 2\partial/\partial r - 2/r^2 + \phi_{P1}(r)]\, \psi_1(r) = 0 \tag{8.3}$$

$$\phi_{P1}(r) = -\tfrac{1}{6}\left[\int_0^r dr'\, \psi_1^2(r')\, r'^2(3r + r'^2/r) + \int_r^\infty dr'\, \psi_1^2(r')\, r'(3r'^2 + r^2)\right] + d_{m1}/r \tag{8.4}$$

where E_{01} is associated with $S=1$ and replaces E_0. These equations are essentially the same as the 0^- equations (7.3) and (7.4) except for the extra

$-2/r^2$ term in (8.3) signifying its $S=1$ nature. Therefore, (7.5) and (7.6) hold with the subscript 0 replaced by 1, except for the right side of (7.5a), which is replaced by $d_1 r$ or $d'_1 r^{-2}$, and the redefined d_1 are similar nonfree constants. By making the same assumption as those preceding (7.7), it is seen that

$$E_{01}^2/4 - M_m^2 = -(d_{m1}/2(n_{r1}+2))^2 \tag{8.5}$$

where n_{r1} denotes the radial quantum number.

Similar to the $S=0$ case, $\phi_P(\mathbf{x})$ of (8.1) is also independent of the angles in the $r \to 0$ and ∞ limits and takes a form analogous to (7.11):

$$\phi_P(\mathbf{x} \to 0) \approx -\frac{1}{8\pi} \int d^3\mathbf{x}\, |\mathbf{x}|\, \mathrm{Re}(\psi(-\mathbf{x})\, \chi^*(-\mathbf{x})) + d_{m1}/r \tag{8.6a}$$

$$\phi_P(\mathbf{x} \to \infty) \approx -\frac{1}{8\pi} r \int d^3x\, \mathrm{Re}(\psi(-\mathbf{x})\, \chi^*(-\mathbf{x})) = -\beta_{m1l} r \tag{8.6b}$$

In these limits, the generally nonseparable (6.8) can be separated and reduced to one-dimensional equations. These and their solutions are given in Appendix B.

Should solutions for all r exist for these in effect sixth-order equations, (B5) shows that there exist several solutions for given l and radial wave number. This calls for a particle classification different from the usual nonrelativistic one (Particle Data Group, 1990).

9. MODEL FOR INTERNAL FUNCTION AND MASS OPERATOR

In the above sections, M_m^2 has been regarded as a constant, formally obtained from (5.5). To determine its value, a simple model of the internal meson function $\zeta^p{}_r(z_{\mathrm{I}}, z_{\mathrm{II}})$ and mass operator m_{2op} of (5.5) is considered below. For simplicity, let $z_{\mathrm{I}} \equiv z$, $\partial/\partial z_{\mathrm{I}} \equiv \partial_z$, $z_{\mathrm{II}} \equiv u$, and $\partial/\partial z_{\mathrm{II}} \equiv \partial_u$. The simplest form of internal quark functions is $\xi_{\mathrm{A}}^p(z_{\mathrm{I}}) = z^p$ and $\xi_{\mathrm{B}r}(z_{\mathrm{II}}) = u_r$ conventionally adopted in the literature. Their product $z^p u_r$ is the corresponding meson internal function. Since z_{I} and z_{II} are in principle not observable, an interchange among them cannot be detected. Thus, the total quark wave function of Section 5 can for instance be $\chi_{\mathrm{A}\dot{c}}(x_{\mathrm{I}})\, \xi_{\mathrm{A}}^p(z_{\mathrm{I}})$ or $\chi_{\mathrm{A}\dot{c}}(x_{\mathrm{I}})\, \xi_{\mathrm{A}}^p(z_{\mathrm{II}})$. This degree of freedom allows an internal symmetry so that not only $z^p u_r$, but also $z_r u^p$ may be associated with the same meson. The right side of (5.3a) can thus be $\zeta_r^p(z_{\mathrm{II}}, z_{\mathrm{I}})$ also. Thus, the simplest generalized internal meson functions symmetric and antisymmetric in z and u are

$$\xi_S{}^p{}_r(z_{\mathrm{I}}, z_{\mathrm{II}}) = \xi_S{}^p{}_r = z^p u_r + z_r u^p = \xi_{Sr}{}^p \tag{9.1a}$$

$$\xi_{\mathrm{A}}{}^p{}_r(z_{\mathrm{I}}, z_{\mathrm{II}}) = \xi_{\mathrm{A}}{}^p{}_r = z^p u_r - z_r u^p = -\xi_{\mathrm{A}r}{}^p \tag{9.1b}$$

The associated internal index symmetry mirrors the space-time index symmetry of (6.1b). For agreement with baryon data, it is postulated in Section 4 of Hoh (n.d.-a) that the total hadron wave function must be symmetric under simultaneous interchange of any two pairs of quark indices. This is equivalent to the assumption of the so-called symmetric quark model and implies that

$$\psi^{a}_{\dot{b}}(x_{\rm I}, x_{\rm II})\, \xi^{p}{}_{r}(z_{\rm I}, z_{\rm II}) = \psi^{b}_{\dot{a}}(x_{\rm I}, x_{\rm II})\, \xi_{r}{}^{p}(z_{\rm I}, z_{\rm II}) \tag{9.2}$$

Combining (9.1), (9.2), and (6.1b), it is seen that the total $S=0$ and $S=1$ meson wave functions are $\psi'_0 \xi^{p}_{\mathrm{A}r}$ and $\psi' \xi^{p}_{Sr}$, respectively. The postulate renders these to be unique by excluding the antisymmetric combinations, much like the limiting effect of Pauli's theorem for identical particles in conventional quantum mechanics.

Inserting the simplified quark internal functions into (5.1) and (5.2) and identifying with (4.5) and (4.6) leads to

$$m_{\mathrm{A}op} = \sum m_s (z^s \partial_{zs} + u^s \partial_{us}) \tag{9.3a}$$

$$m^{*}_{\mathrm{B}op} = \sum m_s (z_s \partial^{s}_{z} + u_s \partial^{s}_{u}) \tag{9.3b}$$

where $m_{\mathrm{A}} \to m_p$ and $m^{*}_{\mathrm{B}} \to m_r$ may be called the bare quark masses having flavor p and antiflavor r and

$$\begin{aligned} \partial_{zs} &= \partial/\partial z^s = \partial/\partial z^{s}_{\rm I}, & \partial^{s}_{z} &= (\partial_{zs})^* \\ \partial_{us} &= \partial/\partial u^s = \partial/\partial z^{s}_{\rm II}, & \partial^{s}_{u} &= (\partial_{us})^* \end{aligned} \tag{9.4}$$

Here z and u may be regarded as creation operators standing to the left of ∂_z and ∂_u considered to be annihilation operators, in agreement with normal ordering in field theory. From (5.3b) and (9.3), the following general form can be written:

$$m_{2op} = m_{2op}(z^p, u^p, z_r, u_r, \partial_{zp} + \partial_{up}, \partial^{r}_{z} + \partial^{r}_{u}, m_p, m_r) \tag{9.5}$$

Equations (9.3), (5.1), and (5.2) also indicate that m_s is associated with the annihilation operators and enters in the form of $m_s \partial_{zs}$ and $m_s \partial_{us}$ and their complex conjugates. One of the simplest forms of (9.5) having dimension mass squared is

$$m_{2op}\binom{p}{r} = \left[\frac{1}{2} m_p (z^p \partial_{zp} + u^p \partial_{up}) + \frac{1}{2} m_r (z_r \partial^{r}_{z} + u_r \partial^{r}_{u})\right]^2 \tag{9.6a}$$

where the bracketed terms can be said to be half of a bare quark mass summing operator. When applied to (9.1), (9.6a) yields the eigenvalue

$$M_m^2\begin{pmatrix}p\\r\end{pmatrix}=\frac{1}{4}(m_p+m_r)^2 \tag{9.6b}$$

Equation (9.1), however, does not represent π^0, ρ^0, ω, η, η', ..., mesons which involve two and three quark–antiquark pairs. Since the space-time part is the same for different flavors, only (9.1) and (9.6a) need modification. Thus,

$$\xi_S{}^p{}_p=z'u_1+z_1u'\pm(z^z u_z+z_z u^z) \tag{9.7}$$

is assigned to the ω (upper sign) and ρ^0 (lower sign) mesons. For these states, the generalization procedure of Section 5 implies that m_{2op} depends upon all the quarks entering (9.7). As (9.7) possesses an additional symmetry in the interchange of the indices 1 and 2 and is furthermore invariant under complex conjugation, the general form of m_{2op} is allowed to possess similar properties. The corresponding simplest form of m_{2op} having (9.7) as eigenfunction can be written as

$$\begin{aligned}m_{2op}\begin{pmatrix}1&\pm&2\\1&\pm&2\end{pmatrix}&=\frac{1}{2}m_{2op}\begin{pmatrix}1\\1\end{pmatrix}+\frac{1}{2}m_{2op}\begin{pmatrix}2\\2\end{pmatrix}_{Q=0}\\&\quad\pm\frac{1}{2}\left[m_1^2(z^2\partial_{z1}+u^2\partial_{u1})(z_2\partial_z^1+u_2\partial_u^1)\right.\\&\quad\left.+m_2^2(z^1\partial_{z2}+u^1\partial_{u2})(z_1\partial_z^2+u_1\partial_u^2)\right]\end{aligned} \tag{9.8a}$$

having the eigenvalues

$$M_m^2\begin{pmatrix}1&\pm&2\\1&\pm&2\end{pmatrix}=\frac{1}{2}(m_1^2+m_2^2) \tag{9.8b}$$

If the two indices become one index, the extra symmetrizing terms in the brackets of (9.8a) have to be excluded and (9.8a) reduces to (9.6a) with $Q=0$. Equation (9.7) with $u^p\to -u^p$ is assigned to the π^0 meson and an unobserved 0^- internal isosinglet, here denoted by p^0 (lower sign). When applied to by (9.8a), (9.8b) is again obtained.

The η-meson internal function is

$$\xi_A{}^p{}_p(\eta)=z^1u_1-z_1u^1+z^2u_2-z_2u^2-2z^3u_3+2z_3u^3 \tag{9.9a}$$

For a simplified treatment, let the indices 1 and 2 collapse into one index 1; (9.9a) then has the same form as (9.7) with the lower sign, equations

(9.8) with index $2 \to 3$ then apply. Let now the index 1 component be split back into index 1 and 2 components and repeat the procedure from (9.7), with the upper sign, to (9.8); one finds

$$M_m^2(\eta) = \tfrac{1}{4}(m_1^2 + m_2^2 + 2m_3^2) \tag{9.9b}$$

The η'-meson internal function is

$$\xi_{\mathrm{A}}{}^{p}{}_{p}(\eta') = \sum_{s=1}^{3} (z^s u_s - z_s u_{\dot{s}}^{s}) \tag{9.10}$$

Equation (9.8a) is now generalized to include a third index in the form of the additional term $m_{2op}(\frac{3}{3})/2$ and four more terms in the brackets to include the combinations of indices 1 and 3 and of 2 and 3. When applied to (9.10), it yields the eigenvalue

$$M_m^2(\eta') = \tfrac{1}{2}(m_1^2 + m_2^2 + m_3^2). \tag{9.11}$$

10. APPLICATION AND DISCUSSION

In the following, electroweak effects as well as the difference of the v and d quark masses will be neglected. Consider first the $J^P = 0^-$ and 1^- mesons. Their space-time parts are accounted for by (7.3), (7.4), (8.3), and (8.4). Dividing (7.3) by $\psi_0(r)$, it is seen that the eigenvalue $E_{00}^2/4 - M_m^2$ depends only upon $\psi_0(r)$ and some constants. A similar statement holds for (8.3). The difference between these two expressions can be written as

$$E_{01}^2 - E_{00}^2 = L_1(n_{r1}) - L_0(n_{r0}) \tag{10.1}$$

where L_0 and L_1 are functions of the radial quantum numbers. The 0^- and 1^- splitting arises naturally from the pseudoscalar interaction among the quarks without recourse to the hyperfine splitting mechanism in QCD inspired type of treatments [2.3].

A prediction is that all the 0^- and 1^- mesons have the same radial wave functions $\psi_0(r)$ and $\psi_1(r)$, respectively, independent of their quark content. The strong-interaction radius of all pseudoscalar mesons is the same. A similar statement holds for the vector mesons. Another prediction is the lack of unexcited 0^+ and 1^+ states mentioned before (6.10) and (8.3), in agreement with data.

Equation (10.1) agrees with data for the $n_r = 0$ and flavored mesons, i.e., π, K, D, D_s, $B(0^-)$ and ρ, K^*, D_s, D_s^*, $B^*(1^-)$ with $L_1(0) - L_0(0) \approx 0.56\ \mathrm{GeV}^2$ (Lichtenberg, 1987), in spite of the large difference between the squares of the π and B masses. For the flavorless J/ψ

and η_c, the somewhat higher value 0.71 GeV^2 indicates that η_c is not a pure $c\bar{c}$ state like J/ψ, but can contain up and down quarks, analogous to η' and η. The absence of the 0^- singlet p^0 and the 0^- $s\bar{s}$, $x\bar{x}$, and $b\bar{b}$ states has been suggested as a consequence of $U(1)$ gauge invariance (Hoh, n.d.-b) of (5.4) together with (5.5).

Equations (9.8b) and (9.6b) show that π^0 is slightly heavier than $\pi^\pm$. However, the difference, for usual bare quark masses (Lichtenberg, 1987) is small compared to the greater but neglected electromagnetic self-energy. Equations (9.6b), (9.9b), (9.11), and (7.3) predict that $E_{00}(\eta') > E_{00}(\eta) > E_{00}(K)$. However, the differences, assuming usual bare quark masses, are smaller than those of the data.

Consider the lower limit equations (7.7) and (8.5) and take the difference. Making use of (10.1) for the $n_r = 0$ and flavored mesons above, one finds

$$d_{m0}^2 - d_{m1}^2/4 = L_1(0) - L_0(0) \approx 0.56 \text{ GeV}^2 \tag{10.2}$$

Assuming further that the d's are equal, (10.2) yields the estimate $d_{m1} = d_{m0} = 0.864$ GeV. Inserting this value and the measured π, K, D, D_s, and B masses E_{00} into (7.7) and combining it with (9.6b), neglecting the Q's, yields the upper limits of the bare quark masses $m_1 = m_2 = 0.437$, $m_3 = 0.559$, $m_4 = 1.62$, $m_4 = 1.591$ (from D_s), and $m_5 = 4.912$. The unit is GeV. Repeat the same procedure for ω, φ, J/ψ, and Υ and replace (7.7) by (8.5). The upper limits of the bare quark masses are found to be $m_1 = m_2 = 0.447$, $m_3 = 0.554$, $m_4 = 1.563$, and $m_5 = 4.735$. Both sets of values are relatively close and may support the assumption $d_{m0} = d_{m1}$ above and that the Q's are small. These data also agree approximately with the bare quark masses estimated in an analogous manner from baryon data in Section 9 of Hoh (n.d.-a).

Existence of unique discrete eigenvalue type of solutions to (7.3), (7.4), (8.3), and (8.4) has not been proven. Assuming the above estimates of d_m, these equations are solved numerically by an iterative procedure. Eigenvalue solutions satisfying the prescribed boundary conditions appear to converge to unique forms after a number of iterations. Two such solutions are shown in Figs. 1 and 2 for $m_1 = m_2 = 0.323$ in (9.8b), making use of the π^0 mass. This value is less than 0.437 obtained below (10.2). The difference is due to the omission of the confining terms in (7.4) and (8.4), corresponding to ϕ'_{P0} and ϕ'_{P1} in Figs. 1 and 2. For the cases represented by these figures, the potential terms in (7.4) and (8.4) can be averaged over r weighted by $r^2\psi_0^2(r)$ and $r^2\psi_1^2(r)$, respectively. The ratio of the confining term to the Coulomb-like term in (7.4) for 0^- slightly exceeds unity. The corresponding ratio in (8.4) for 1^- is about 2/3. Further, the positive eigenvalue

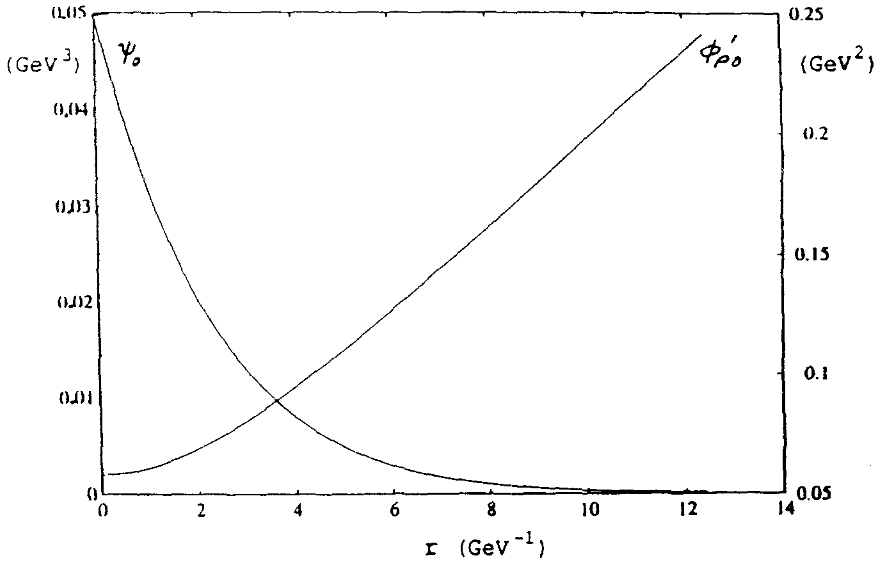

Fig. 1. Radial wave function and confining part of potential for pseudoscalar mesons obtained from (7.3) and (7.4) for $d_{m0} = 0.8641$ GeV and $E_{00}^2/4 - M_m^2 = -0.1$ GeV^2 corresponding to a bare quark mass of $m_1 = m_2 \approx M_m \approx 0.323$ GeV. $-\phi'_{P0} = \phi_{P0} - d_{m0}/r$.

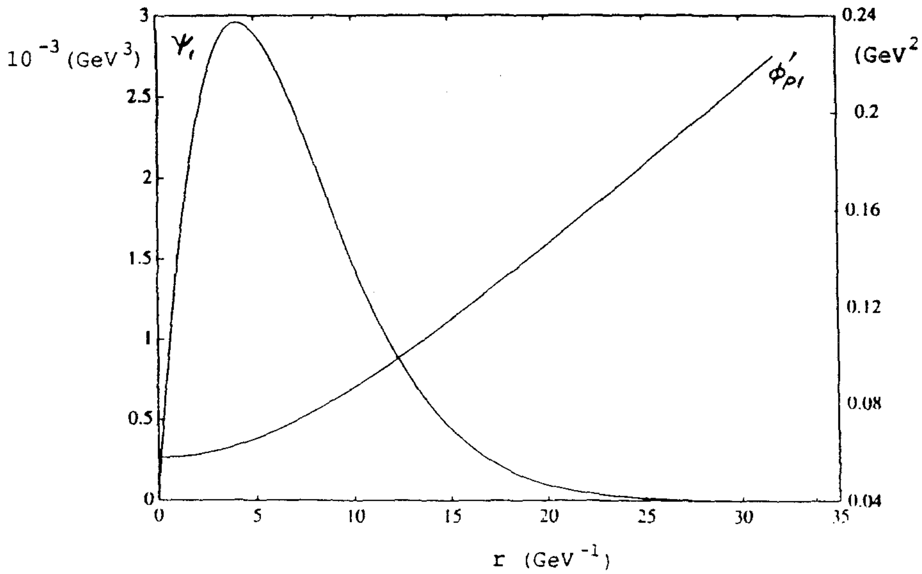

Fig. 2. Radial wave function and confining part of potential for vector mesons obtained from (8.3) and (8.4) for $d_{m1} = 0.8641$ GeV and $E_{01}^2/4 - M_m^2 = 0.04$ GeV^2 corresponding to a bare quark mass of $m_1 = m_2 \approx M_m \approx 0.323$ GeV. $-\phi'_{P1} = \phi_{P1} - d_{m1}/r$.

$E_{01}^2/4 - M_m^2$ in Fig. 2 becomes negative when the confining term in (8.4) is dropped in the approximation leading to (8.5). Therefore, the confining term plays an additional basic role here.

Let the strong interaction radii $\langle r_0 \rangle$ and $\langle r_1 \rangle$ of the pseudoscalar and vector mesons be defined by $\psi_0(\langle r_0 \rangle)/\psi_0(0) = 1/\sqrt{2}$ and $\psi_1(\langle r_1 \rangle) =$ maximum of $\psi_1(r)$, respectively. Figures 1 and 2 show that $\langle r_0 \rangle = 0.97$ fm and $\langle r_1 \rangle = 5.1$ fm. Another calculation with $m_1 = m_2 = 0.393$ yields $\langle r_0 \rangle = 1.0$ fm. Thus, $\langle r_0 \rangle$ seems to be rather insensitive to the bare quark mass. Experimentally, the strong interaction radius is neither well defined nor well determined, but is of the same magnitude as the better determined electromagnetic radius. The present values are nevertheless somewhat high, but can be reduced if d_{m0} and d_{m1} exceed 0.864.

The radially excited states such as $\psi(2S)$, $\Upsilon(2S), \ldots$, can be associated with $L_1(n_{r1} > 0)$. Similarly, $L_0(n_{r0} > 0)$ are assigned to radially excited 0^- states. For the angularly excited states, the nonseparability of the $S = 0$ equations (6.10) and (7.2) in r and ϑ equations prevents the usual nonrelativistic l (orbital quantum number) and n (total quantum number) classification of mesons. Should solutions exist, they may be assigned to the b_1, π_2, $K_1, \ldots$ mesons (Particle Data Group, 1990).

The nonseparable $S = 1$ equations (6.8) and (8.1) for angularly excited states differ from the corresponding $S = 0$ ones above and the vector meson equations of Section 8 in that they have more component wave functions. Assuming that a lowest angularly excited solution exists, then the r–ϑ coupling again prevents the usual nonrelativistic particle classification. The present classification is indicated by the solution of these equations near $r = 0$; (B5) shows that there are three sets of solutions regular at $r = 0$ associated with fr^2, hr, and g, which in turn are associated with Y_2, Y_1, and Y_0, respectively, in (B1) and (B2). These states fit naturally the observed triplets a_ν, the lightest f_ν, $\chi_{c\nu}(1P)$, and $\chi_{b\nu}(1P)$, where $\nu = 1$, 2, and 3 (Particle Data Group, 1990). The triplet $\chi_{b\nu}(2P)$ and the next heavier f_ν may then be assigned to the radially excited version of these states.

The members of the triplet may be regarded as different modes of the same so-called particle, contrary to different particles as in nonrelativistic classification schemes.

In Section 6, it was shown that

$$E_0^2(\mathbf{K}) = \mathbf{K}^2 + E_0^2 \tag{10.3}$$

where $E_0(\mathbf{K})$ denotes the total meson energy, can be fulfilled at $\mathbf{K} = 0$ and ∞. At $\mathbf{K} = 0$, the relative energy ω_0 can not be determined in the present model but is required to vanish in a quantized treatment [9]. In the $\mathbf{K} = \infty$ limit, (6.12b) shows that $\omega_0 = \pm|\mathbf{K}|$ is possible. Therefore, it may be conjectured that ω_0 assumes such values that (10.3) holds.

The $\mathbf{K} \to \infty$ limit considered at the end of Section 6 can also be viewed from another viewpoint. Equation (4.11), dropping its right sides, may via the inverse of (4.8) be decomposed into two massless and free fermions. This is in agreement with the well-known phenomenon of asymptotic freedom.

APPENDIX A. SOME NOTATIONS AND DEFINITIONS

x_{I} represents the space-time coordinates $x_{\mathrm{I}}^{\mu} = (x_{\mathrm{I}}^0, \mathbf{x}_{\mathrm{I}}) = (x_{\mathrm{I}}^0, x_{\mathrm{I}}^1, x_{\mathrm{I}}^2, x_{\mathrm{I}}^3)$. $\partial_{\mathrm{I}\mu} = \partial/\partial x_{\mathrm{I}}^{\mu}$. Further,

$$\Box_{\mathrm{I}} = \partial^2/\partial(x_{\mathrm{I}}^0)^2 - \partial^2/\partial\mathbf{x}_{\mathrm{I}}^2 \tag{A1}$$

$$\begin{aligned}
\partial_{\mathrm{I}}^{1\dot{1}} &= \partial_{\mathrm{I}\dot{2}2} = -\partial_{\mathrm{I}0} - \partial_{\mathrm{I}3} \\
\partial_{\mathrm{I}}^{2\dot{2}} &= \partial_{\mathrm{I}\dot{1}1} = -\partial_{\mathrm{I}0} + \partial_{\mathrm{I}3} \\
\partial_{\mathrm{I}}^{1\dot{2}} &= -\partial_{\mathrm{I}\dot{1}2} = -\partial_{\mathrm{I}1} + i\partial_{\mathrm{I}2} \\
\partial_{\mathrm{I}}^{2\dot{1}} &= -\partial_{\mathrm{I}\dot{2}1} = -\partial_{\mathrm{I}1} - i\partial_{\mathrm{I}2}
\end{aligned} \tag{A2}$$

These definitions also hold for the subscript $\mathrm{I} \to \mathrm{II}$.

Dirac's bispinor ψ_{A} and van der Waerden's spinors ψ_{A}^{a} and $\chi_{\mathrm{A}\dot{b}}$ are related by

$$\begin{aligned}
\chi_{\mathrm{A}\dot{1}} &= (\psi_{\mathrm{A}})_1 + (\psi_{\mathrm{A}})_3, \qquad & \chi_{\mathrm{A}\dot{2}} &= (\psi_{\mathrm{A}})_2 + (\psi_{\mathrm{A}})_4 \\
\psi_{\mathrm{A}}^1 &= (\psi_{\mathrm{A}})_1 - (\psi_{\mathrm{A}})_3, \qquad & \psi_{\mathrm{A}}^2 &= (\psi_{\mathrm{A}})_2 - (\psi_{\mathrm{A}})_4
\end{aligned} \tag{A3}$$

The same holds for $\mathrm{A} \to \mathrm{B}$.

APPENDIX B. $S=1$ EQUATIONS AT $r \to 0$ AND ∞

$\boldsymbol{\psi}(\mathbf{x})$ and $\boldsymbol{\chi}(\mathbf{x})$ are expanded into vector spherical harmonics [e.g., Blatt and Weisskopf (1979), Appendix B]. There are three kinds of such harmonics, each consisting of three series of products of $Y_{lm}(\vartheta, \varphi)$ and Clebsch–Gordan coefficients. For given l and m, $\boldsymbol{\psi}'(\mathbf{x}) = (\psi^1, \psi^2, \psi^3)$ and $\boldsymbol{\chi}'(\mathbf{x}) = (\chi^1, \chi^2, \chi^3)$ are found to be

$$\begin{aligned}
(\psi^1 \mp \psi^2) + (\chi^1 \pm \chi^2) &= \mp\left(\frac{(l\mp m+1)(l\mp m+2)}{l(2l+3)}\right)^{1/2} Y_{l+1\,m\mp 1} f(r) \\
&\quad \pm\left(\frac{(l\pm m-1)(l\pm m)}{l(2l-1)}\right)^{1/2} Y_{l-1\,m\mp 1} g(r) \\
\psi^3 + \chi^3 &= \left(\frac{(l-m+1)(l+m+1)}{l(2l+3)}\right)^{1/2} Y_{l+1\,m} f(r) \\
&\quad + \left(\frac{(l-m)(l+m)}{l(2l-1)}\right)^{1/2} Y_{l-1\,m} g(r)
\end{aligned} \tag{B1}$$

$$(\psi^1 \mp \psi^2) - (\chi^1 \mp \chi^2) = i\left(\frac{(l \pm m)(l \mp m + 1)}{l(l+1)}\right)^{1/2} Y_{lm \mp 1} h(r)$$

$$\psi^3 - \chi^3 = -i \frac{m}{[l(l+1)]^{1/2}} Y_{lm} h(r) \tag{B2}$$

$\psi'(\mathbf{x})$ and $\chi'(\mathbf{x})$ are eigenfunctions of the $\mathbf{J}^2$, J_3, and $\mathbf{S}^2$ with eigenvalues $l(l+1)$, m, and 2, respectively. Here $\mathbf{J}$ denotes the total angular momentum operator, J_3 its third component, and $\mathbf{S}$ the spin operator. For $l=0$, only one component $Y_1 f(r)$ exists corresponding to (8.2). For $l>0$, (B1), (B2), and (6.8) can be separated into Y_{l+1}, Y_l, and Y_{l-1} components associated, respectively, with the following radial equations:

$$\begin{aligned}
&\left[-\frac{E_0^2}{4} + M_m^2 - \phi_P(\mathbf{x} \to {}^0_\infty) - \frac{1}{2l+1} \Delta_{l+1}\right] f(r) \\
&\quad - \frac{2l}{2l+1}\left[\frac{\partial^2}{\partial r^2} - \frac{2l-1}{r}\frac{\partial}{\partial r} + \frac{(l-1)(l+1)}{r^2}\right] g(r) \\
&\quad - E_0 \frac{l}{2l+1}\left(\frac{\partial}{\partial r} - \frac{l}{r}\right) h(r) = 0 \\
&\left[\frac{E_0^2}{4} + M_m^2 - \phi_P(\mathbf{x} \to {}^0_\infty) - \Delta_l\right] h(r) \\
&\quad - E_0\left(\frac{\partial}{\partial r} + \frac{l+2}{r}\right) f(r) + E_0\left(\frac{\partial}{\partial r} - \frac{l-1}{r}\right) g(r) = 0 \\
&\left[-\frac{E_0^2}{4} + M_m^2 - \phi_P(\mathbf{x} \to {}^0_\infty) + \frac{1}{2l+1} \Delta_{l-1}\right] g(r) \\
&\quad - \frac{2l+2}{2l+1}\left[\frac{\partial^2}{\partial r^2} + \frac{2l+3}{r}\frac{\partial}{\partial r} + \frac{l(l+2)}{r^2}\right] f(r) \\
&\quad + E_0 \frac{l+1}{2l+1}\left(\frac{\partial}{\partial r} + \frac{l+1}{r}\right) h(r) = 0
\end{aligned} \tag{B3}$$

$$\Delta_l = \frac{\partial^2}{\partial r^2} + \frac{2\partial}{\partial r} - \frac{l(l+1)}{r^2} \tag{B4}$$

These radial equations are sixfold degenerate, i.e., the six component equations of (6.8) yield the same radial equation.

The standard treatment (Coddington and Levinson, 1985, Chapters 4 and 5) of (B3) is to convert it into six first-order equations and consider the $r \to 0$ and ∞ behaviors there. There are now six independent solutions. For $r \to 0$, these are

$$\begin{aligned} &f=f_+r^{l+1}, && g=(l+1)(2l+3)f, && h=0 \\ &f=f_-r^{-l}, && g=f/l(2l-1), && h=0 \\ &f=g=0, && h=h_+r^l, && h_-r^{-l-1} \\ &f=h=0, && g=g_+r^{l-1}, && g_-r^{-l-2} \end{aligned} \tag{B5}$$

where $f_\pm$, $g_\pm$, and $h_\pm$ are constants. In the $r\to\infty$ limit,

$$\begin{aligned} &f=g=f_\infty r^\tau e^{-\eta} \\ &h=h_\infty r^{\tau-1/2}e^{-\eta} \\ &\eta=2(\beta_{m1l}r)^{3/2}/3+\alpha_{m1}r^{1/2} \end{aligned} \tag{B6}$$

where f_∞, h_∞, τ, and α_{m1} are constants. Possible logarithmic terms (Coddington and Levinson, 1955, Chapters 4 and 5) in (B6) are not included here.

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